On generalized sequential spaces

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Abstract

Maps from well-ordered spaces are employed to develop a transfinite sequence $\mathbf{TOP}_J$ of cartesian closed categories of spaces lying strictly between the category of sequential spaces (**SEQ**) and the category of compactly generated spaces (**CG**). Familiar characterizations of "weakly Hausdorff" within **SEQ** generalize naturally to $\mathbf{TOP}_J$ and the k-product $X \times_k Y$ of **CG** is preserved within $\mathbf{TOP}_J$.

1 Introduction

We employ maps of well-ordered spaces to generalize the notion of sequential space and develop a transfinite sequence of Cartesian closed categories $\mathbf{TOP}_J$, each of which contains the sequential category **SEQ** and is contained in both the category (**CG**) of compactly generated spaces and (**TF**) transfinite sequential spaces [17]. The weakly Hausdorff spaces of $\mathbf{TOP}_J$ are precisely those which enjoy "J-unique convergence," a natural generalization of unique sequential convergence.

To motivate this paper we recall a few definitions. The space X is *sequential* if the closed sets of X are determined by convergent sequences. Specifically A is not closed in X iff there exists in X a convergent sequence $x_1, x_2, ... \to x$ such that $x_n \in A$ but $x \notin A$. Every metric space is sequential and, more generally, the sequential spaces are precisely the quotients of metric spaces [11]. Sequential spaces continue to play a useful role in the theory of computation [23][24].

A Hausdorff space X is compactly generated (or equivalently a *k-space*; see [2][18][25][29]) if the closed sets of X are precisely the sets which have compact intersection with each compact subspace of X. This definition becomes problematic for non-Hausdorff spaces and we employ the modern notion of *compactly generated* [3][26]: for each non-closed $A \subset X$ there exists a compact Hausdorff space K and a map $f : K \to X$ such that $f^{-1}(A)$ is not compact. Compactly generated spaces form a convenient, Cartesian closed category **CG** with respect to the internal categorical k-product $\times_K$: See [1][25][26]. Given compactly generated spaces X and Y, the topology of $X \times_K Y$ is the coarsest refinement of the standard product topology such that $X \times_K Y$ is compactly generated.

The notion of *transfinite sequential space* generalizes that of sequential space by replacing convergent sequences with convergent nets indexed by limit ordinals

[17]. Thus in a transfinite sequential space X the closed subsets are determined by convergent nets indexed over limit ordinals as follows. If $A \subset X$, then A is not closed in X iff there there exists a limit ordinal J and a function $f : J \cup \{\infty_J\} \to X$ from its successor such that $f(J) \subset A$, $f(\infty_J) \notin A$, and such that f is "continuous at ∞_J" (provided $J \cup \{\infty_J\}$ has the order topology). Since f is not required to be globally continuous, a transfinite sequential space X can fail to be compactly generated as shown in Example 34. Thus we seek to strengthen transfinite sequential spaces, creating a class of compactly generated spaces whose topology is determined by globally continuous maps of well-ordered sets with the order topology.

Fixing a limit ordinal J with successor $J \cup \{\infty_J\}$ (both with the order topology) a subset $I \subset J$ is *almost closed* in $J \cup \{\infty_J\}$ if I contains all of its limit points except for $\ell = \min\{j \in J | j > I\}$. In a manner similar to sequential spaces, we declare X to be a *J-convergence space* if for each non-closed $A \subset X$ there exists an almost closed set I and a map $f : I \cup \{\ell\} \to X$ such that $f(I) \subset A$ and $f(\ell) \notin A$, and we declare $\mathbf{TOP}_J$ to be the category of J-convergence spaces. Since compact ordinals are Hausdorff it follows immediately that every J-convergence space is compactly generated, i.e. $\mathbf{TOP}_J \subset \mathbf{CG}$. On the other hand, since the limit ordinal $\{1, 2, 3, ...\}$ is almost closed in $J \cup \{\infty_J\}$, we have $\mathbf{SEQ} \subset \mathbf{TOP}_J$.

The internal categorical product $\times_J$ of $\mathbf{TOP}_J$ is similar to the k-product in $\mathbf{CG}$; the topology of $X \times_J Y$ is the coarsest refinement of the product topology ensuring the space is a J-convergence space. We use this product to show $\mathbf{TOP}_J$ is a convenient, Cartesian closed category (Theorem 23). Additionally, given spaces X and Y in $\mathbf{TOP}_J$, and underlying set $X \times Y$, the topologies of $X \times_J Y$ and $X \times_K Y$ both refine the standard product topology and guarantee the product spaces are (respectively) in $\mathbf{TOP}_J$ and $\mathbf{CG}$. Theorem 36 ensures $X \times_J Y = X \times_K Y$. The critical step (Lemma 22) in both Cartesian closedness and the agreement of categorical products is to show the product $J \cup \{\infty_J\} \times J \cup \{\infty_J\}$ (with the usual product topology) is a J-convergence space. A consequence of these categorical properties is that standard facts about $\mathbf{CG}$ translate nicely into $\mathbf{TOP}_J$. In particular, $\mathbf{TOP}_J$ is well behaved with respect to products of quotient maps (Corollary 26).

The space X is *weakly Hausdorff* if all maps from Hausdorff compacta into X are closed maps. The category $\mathbf{CGWH}$ of compactly generated weakly Hausdorff spaces is, for many purposes, better behaved than $\mathbf{TOP}$ with regard to various standard constructions such as products, quotients and colimits [22]. For example, the errata to [19] (8. p. 485 [20]) notes $\mathbf{CGWH}$ is the correct category for [19].

A familiar fact in $\mathbf{SEQ}$ is the following: the sequential space X is weakly Hausdorff iff each convergent sequence in X has a unique limit (i.e. if X is a US-space) [28]. Example 44 suggests there is no corresponding result in $\mathbf{TF}$, however, we find a natural analogue in $\mathbf{TOP}_J$. Given a limit ordinal J, we declare X is a "J-unique convergence space" if each (continuously extendable) map $f : J \to X$ has a unique extension to $J \cup \{\infty_J\}$. Theorem 43 shows if $X \in \mathbf{TOP}_J$ then X is weakly Hausdorff iff X is a J-unique convergence space.

Throughout the paper we frequently employ the notion of "KC-space" (a space whose compacta are closed [12][28]). The KC-property is stronger than the weak Hausdorff property, but the two notions are equivalent in **CG**.

This paper is structured as follows. In Section 2, we recall some basic definitions and properties pertaining to well-ordered spaces. In section 3, we define the notion of J-convergence space, study the general properties of such spaces, and prove the category of J-convergence spaces is Cartesian closed. We end section 3 comparing J-convergence spaces with compactly generated spaces and transfinite sequential spaces as defined previously in the introduction. In section 4, we define the notion of a J-unique space and include a proof of Theorem 43.

2 Well-ordered spaces

A linearly ordered set J is **well-ordered** if for each subset $A \subset J$, $\min(A)$ exists. A **well-ordered space** is a well-ordered set with the **order topology** generated by a basis consisting of intervals $(a, b] = \{j \in J | a < j \leq b\}$.

Remark 1. Suppose J is a well-ordered space with minimal element 0 and $|J| > 1$. The topology of J is generated by subbasic sets of the form $[0, j)$ and $(j, \infty) = \{j' \in J | j' > j\}$ for $j > 0$. Note, if J is compact, there is a maximal element $m > 0$ and so $(j, m] = (j, \infty)$.

If X is a non-compact space, the **Alexandroff compactification of** X is the space $X \cup \{\infty_X\}$, obtained by adjoining a single point ∞_X, whose open sets are precisely the sets $U \subset X$ such that U open in X and sets $(X \cup \{\infty_X\}) \backslash C$ such that C is closed and compact in X. Note the inclusion $X \to X \cup \{\infty_X\}$ is an embedding.

If J is a non-compact, well-ordered space, the one-point compactification $J \cup \{\infty_J\}$ is Hausdorff (since J is locally compact Hausdorff) and well-ordered when it inherits the ordering of J and ∞_J is maximal. Thus the order type of $J \cup \{\infty_J\}$ is that of the successor of J.

2.1 Almost closed sets

If I is a subset of the well-ordered space S, there are two competing topologies to impart on I, the subspace topology on I, and the internal order topology on I. For example, consider the well-ordered space $S = \{1, 3, 5, ..., 2, 4\}$ and the subset $I = \{1, 3, 5, ...4\} = S \backslash \{2\}$. The subspace topology makes I a non-compact discrete space whereas I with the internal order topology is compact.

Lemma 3 clarifies for which subsets $I \subset S$, the order topology on I is compatible with the subspace topology on I, and yields Corollaries 4 and 5.

Definition 2. Suppose S is a well-ordered space. A subspace $I \subset S$ is **almost closed** in S if (with the subspace topology) $\overline{I} \backslash I = \{\ell\}$ and $\ell = \min\{s \in S | I < s\}$.

Lemma 3. *Suppose $(S, <)$ is a well-ordered space and I is a subset of S. Let $(I, <<)$ denote the space with underlying set I and with order topology generated by the inherited internal order on I. Let $\alpha : (I, <<) \to (S, <)$ denote inclusion. The following are equivalent:*

1. α is continuous.
2. α is an embedding.
3. I is a closed subset of S or I is almost closed in S.

Proof. Let $Y = im(\alpha)$ with the subspace topology. Let i_m be minimal in I and s_m minimal in S.

$1 \Rightarrow 2$. We will show α is an open function onto Y. Suppose U is a basic open set of $(I, <<)$. If $U = (i, j] \cap I$ with $i < j$ and $\{i, j\} \subset I$, then $\alpha(U) = (i, j] \cap I$ and by definition of the subspace topology $\alpha(U)$ is open in Y. Suppose $U = \{i_m\}$. If $i_m = s_m$ then $\alpha(U) = \{s_m\}$ which is open in Y. If $i_m \neq s_m$ then $s_m < i_m$ and hence $\alpha(U) = (s_m, i_m] \cap I$ and thus $\alpha(U)$ is open in Y. Hence $\alpha : (I, <<) \to Y$ is continuous, one to one, and open. Thus α is a homeomorphism onto Y, and hence an embedding into S.

$2 \Rightarrow 3$. To obtain a contradiction, suppose I is neither closed nor almost closed in S. Since I is not closed in S, $\overline{I} \backslash I \neq \emptyset$. Let $l = \min(\overline{I} \backslash I)$.

We will check various cases to show $\{i \in I | l < i\} \neq \emptyset$. First note if I is unbounded in S then $\{i \in I | l < i\} \neq \emptyset$. Henceforth we assume I is bounded in S. If $\left|\overline{I} \backslash I\right| \geq 2$ obtain $l_2 \in \overline{I} \backslash I$ such that $l < l_2$. Since l_2 is a limit point of I, there exists $i \in I$ such that $l < i < l_2$. In particular $\{i \in I | l < i\} \neq \emptyset$.

For the final case we assume $\left|\overline{I} \backslash I\right| = 1$ and I is bounded in S. Thus $\{s \in S | I < s\} \neq \emptyset$. Let $M = \min\{s \in S | I < s\}$. Note $l < M$ (since otherwise I is almost closed in S) and recall $l \notin I$. Hence there exists $i \in I$ such that $l < i < M$ (since otherwise, by definition of M, $M \leq l$). Thus in all cases we have shown $\{i \in I | l < i\} \neq \emptyset$.

Let $m = \min\{i \in I | l < i\}$. Note $m \in Y$ and the open set $(l, m] \cap Y$ shows m is not a limit point of Y. To check that m is a limit point of $(I, <<)$, suppose U is a basic open set in $(I, <<)$ such that $m \in U$. Note $i_m \leq l < m$ and thus we may assume $U = (i, m] \cap I$ for $i < m$ and $i \in I$. Since $l \notin I$, minimality of m ensures $i < l$. Since l is a limit point of Y, there exists $j \in I$ such that $i < j < l$. Hence $\alpha^{-1}(j) \in U$. Thus $\alpha^{-1}(m)$ is a limit point of Y, contradicting the fact that α is a homeomorphism from $(I, <<)$ onto Y.

$3 \Rightarrow 1$. Suppose V is a basic open set of S. Suppose $V = \{s_m\}$. If $i_m = i_s$ then $\alpha^{-1}(V) = \{i_m\}$ which is open in $(I, <<)$. If $i_m \neq i_s$ then $i_s < i_m$ and $\alpha^{-1}(V) = \emptyset$ which is open in $(I, <<)$. Henceforth we assume $V = (a, b]$ with $a < b$ and $\{a, b\} \subset S$. If $(a, b] \cap Y = \emptyset$ then $\alpha^{-1}(V) = \emptyset$ which is open in $(I, <<)$. Henceforth we assume $(a, b] \cap Y \neq \emptyset$.

Working in S, let $y_m = \min(Y \cap (a, b])$ and let $l = \min\{s \in S | \forall y \in (a, b] \cap Y, y \leq s\}$. Note $Y \cap (a, b] = Y \cap [y_m, l]$. Observe, since Y is well-ordered, $y_m \in Y$ and by construction y_m is not a limit point of Y. If $l = y_m = i_m$ then $\alpha^{-1}(V) = \{i_m\}$ which is open in $(I, <<)$.

If $i_m < y_m$ and $y_m = l$ then $\alpha^{-1}(V) = \{y_m\} = (i_m, y_m] \cap I$ which is open in I. Henceforth we assume $y_m < l$.

Let $A = \{y \in Y | y_m < y \leq l\}$. If $i_m = y_m$ let $a_m = i_m$. If $i_m < y_m$ let $a_m = \sup\{y \in Y | y < y_m\}$. Since $[i_m, y_m] \cap Y$ is closed in S, $a_m \in Y$. If $i_m = a_m$ then $\alpha^{-1}(V) = \{i_m\} \cup (\cup_{y \in A}(I \cap (a_m, y]))$. If $i_m < a_m$ then $\alpha^{-1}(V) = \cup_{y \in A}((a_m, y] \cap I)$. In both cases $\alpha^{-1}(V)$ is the union of basic open sets and hence $\alpha^{-1}(V)$ is open in $(I, <<)$. Thus α is continuous. □

Corollary 4. *Suppose S is a well-ordered space and I is almost closed in S. Let $\overline{I} = I \cup \{l\}$. Let $(I, <<)$ denote the space with underlying set I and with order topology generated by the inherited internal order on I. Let $I \cup \{\infty_I\}$ denote the Alexandroff compactification of $(I, <<)$. Define $h : I \cup \{\infty_I\} \to \overline{I}$ such that $h(i) = i$ and $h(\infty_I) = l$. Then h is a homeomorphism.*

Proof. To see that $I \cup \{\infty_I\}$ is well defined we must first show that $(I, <<)$ is not a compact space. By Lemma 3, the restriction $h|_{(I,<<)}$ is a homeomorphism onto I. Since I is not a closed subspace of the Hausdorff space J, I is not compact. To see that h and h^{-1} extend continuously between the respective closures, suppose $U \subset \overline{I}$ and $l \in A$. Since $\overline{I}$ is a compact T_2 space, U is open in $\overline{I}$ iff $\overline{I} \backslash U$ is compact. Hence h is a homeomorphism. □

Corollary 5. *If K is almost closed in $J \cup \{\infty_J\}$ and L is almost closed in $K \cup \{\infty_K\}$, then L is almost closed in $J \cup \{\infty_J\}$.*

Proof. In general, $\overline{A}^X$ denotes the closure of subset $A \subseteq X$ in X. Suppose

$$\overline{L}^{K \cup \{\infty_K\}} \backslash L = \{\ell_L\} \text{ where } \ell_L = \min\{k \in K \cup \{\infty_K\} | L < k\}$$

$$\overline{K}^{J \cup \{\infty_J\}} \backslash K = \{\ell_K\} \text{ where } \ell_K = \min\{j \in J \cup \{\infty_J\} | K < j\}.$$

Since $K \cup \{\infty_K\} \cong \overline{K}^{J \cup \{\infty_J\}}$ (by Corollary 4), $\overline{L}^{K \cup \{\infty_K\}}$ is a closed subspace of $\overline{K}^{J \cup \{\infty_J\}}$ and is therefore closed in $J \cup \{\infty_J\}$. Hence $\overline{L}^{J \cup \{\infty_J\}} = \overline{L}^{K \cup \{\infty_K\}}$ and we have $\overline{L}^{J \cup \{\infty_J\}} \backslash L = \{\ell_L\}$. If there is a $j \in J$ such that $L < j < \ell_L$, then ℓ_L cannot be a limit point of L in J. Thus $\ell_L = \min\{j \in J \cup \{\infty_J\} | L < j\}$. □

3 Generalizing sequential spaces

Fixing a non-compact, well-ordered space J with Alexandorff compactification $J \cup \{\infty_J\}$, we generalize the notion of sequential space as follows.

Definition 6. The space X is a **J-convergence space** if for each nonclosed $A \subset X$ there exists $I \subset J$ such that I is almost closed in $J \cup \{\infty_J\}$ and there exists a map $f : I \cup \{\infty_I\} \to X$ such that $f(I) \subset A$ and $f(\infty_I) \notin A$.

Definition 7. Let X be a space. A subset $C \subseteq X$ is **J-closed** if for every almost closed set I in $J \cup \{\infty_J\}$ and map $f : I \cup \{\infty_I\} \to X$ such that $f(I) \subseteq C$, then $f(\infty_I) \in C$. A set U is **J-open** if $X \backslash U$ is J-closed.

In the case of the discrete space of natural numbers $J = \omega$, we retain the familiar notion of sequentially closed.

Proposition 8. *A subset $A \subseteq X$ is J-closed (J-open) iff for every almost closed subspace I in $J \cup \{\infty_J\}$ and every map $f : I \cup \{\infty_I\} \to X$, $f^{-1}(A)$ is closed (open) in $I \cup \{\infty_I\}$.*

Proof. We show this characterization for J-closed sets holds; the analogous characterization of J-open sets follows immediately.

If A is not J-closed, then there is an almost closed subspace I of $J \cup \{\infty_J\}$ and a map $f : I \cup \{\infty_I\} \to X$ such that $f(I) \subseteq A$ and $f(\infty_I) \notin A$. Note $f^{-1}(A) = I$ is not closed in $I \cup \{\infty_I\}$.

For the converse, suppose I is almost closed in $J \cup \{\infty_J\}$ and $f : I \cup \{\infty_I\} \to X$ is a map such that $f^{-1}(A)$ is not closed in $I \cup \{\infty_I\}$. Let i be the minimal limit point of $f^{-1}(A)$ such that $i \notin f^{-1}(A)$. Note $K = f^{-1}(A) \cap [0, i) \subseteq I$ is closed and cofinal in $[0, i)$ and is therefore an almost closed subspace of $I \cup \{\infty_I\}$. By Corollary 5, K is almost closed in $J \cup \{\infty_J\}$. If g is the restriction of f to $K \cup \{\infty_K\}$, then $f(K) \subseteq A$ and $f(\infty_K) = f(i) \notin A$. Thus A is not J-closed. □

Just as with sequentially open sets, the J-open sets in X form a new topology on X. Since every open set in X is J-open, the new topology is no coarser than the original topology on X. Let $\mathbf{J}X$ denote X with the topology of J-open sets.

Remark 9. A space X is a **J-convergence space** iff $X = \mathbf{J}X$, that is, if every J-open (J-closed) subset is open (closed) in X.

3.1 Restriction to injective maps on ordinals

Theorem 12 characterizes J-convergence spaces. In particular, injective maps from well-ordered spaces determined the topology of a T_1 J-convergence space.

Lemma 10. *Suppose J is a non-compact, well-ordered space, X is a space, and $f : J \to X$ is a (not necessarily continuous) function such that, for all $j \in J$, the set $f^{-1}(f(j))$ is bounded. Then there exists a closed cofinal set $I \subset J$ such that $f|_I$ is one to one. In particular, if f is continuous, then $f|_I$ is continuous.*

Proof. In general, to define a set $I \subset J$ it suffices to define a function $\phi : J \to \{0, 1\}$ and define $I = \phi^{-1}(1)$.

To define such a function $\phi : J \to \{0, 1\}$ it suffices to define $\phi(0)$ (where 0 denotes the minimal element of J) and, to specify a process by which, assuming $\phi|_{[0,j)} : [0, j) \to \{0, 1\}$ has been defined, $\phi(j) \in \{0, 1\}$ is then defined.

Define $\phi(0) = 1$.
Suppose $j \in J$ and $\phi|_{[0,j)} : [0, j) \to \{0, 1\}$ is defined.
Select $\phi(j) \in \{0, 1\}$ such that $\phi(j) = 0$ iff both the following conditions are met:

1. The set $\phi|_{[0,j)}^{-1}(1)$ has a maximal element k_j.
2. There exists $m_j \geq j$ such that $f(k_j) = f(m_j)$.

We now check that I is closed, cofinal, and that $f|_I$ is one to one.

We show $f|_I$ is one to one by supposing it is not and obtaining a contradiction. If $f|_I$ is not one to one, there exists $i \in J \cap I$ such that $\left|f^{-1}(f(i))\right| \geq 2$. Obtain $i \in J \cap I$ minimal such that $\left|f^{-1}(f(i))\right| \geq 2$. By minimality of i, if $k < i$ and $k \in I$ then $f^{-1}(f(k)) = \{k\}$ and hence $f(k) \neq f(i)$. Thus there exists $m \in I$ such that $i < m$ and $f(i) = f(m)$. Note $\phi(m) = 1$ since $m \in I$. Since $i < m$ and $\phi(m) = 1$, there exists $j \in I$ minimal such that $\phi(j) = 1$ and $i < j$. Moreover if $i < m < j$, the minimality of j is violated; thus $j \leq m$. Note that if $l \in [0, j)$ and $\phi(l) = 1$ (i.e. $l \in I$), the minimality of j gives that $l \leq i$. Thus i is maximal in $\phi|_{[0,j)}^{-1}(1)$. Now let $k_j = i$ and recall the definition of $\phi(j)$. Since $\phi(j) \neq 0$, it follows that $f(k_j) \neq f(m_j)$ for all $m_j \geq j$. In particular, $f(i) = f(k_j) \neq f(m)$, contradicts our knowledge that $f(i) = f(m)$. Therefore, $f|_I$ is one to one.

To show that I is closed in J, we suppose I is not closed and obtain a contradiction. If $\overline{I} \backslash I \neq \emptyset$, let j be the minimal element of $\overline{I} \backslash I$. Since $j \notin I$, $\phi(j) = 0$. Let k_j be maximal in $\phi|_{[0,j)}^{-1}(1)$ and note by definition $k_j < j$. Since j is a limit point of I, obtain $i \in (k_j, j] \cap I$. Thus $k_j < i < j$ and $\phi(i) = 1$ (since $i \in I$) contradicting the fact that k_j is maximal in $\phi|_{[0,j)}^{-1}(1)$. Therefore I is closed in J.

To show that I is cofinal in J, we suppose it is not and obtain a contradiction. If I is bounded and closed in J, then there exists a maximal element $i \in I$. Since $f^{-1}(f)(i)$ is bounded, obtain $M \in J$ such that if $M \leq m$, then $f(i) \neq f(m)$. In particular, $i < M$. Note that if $j > i$, then $\phi(j) = 0$. Thus i is the maximal element of $\phi|_{[0,M)}^{-1}(1)$. On the other hand, the fact that $\phi(M) = 0$ means there exists $m \geq M$ such that $f(i) = f(m)$, contradicting our choice of M. $\square$

We observe the following characterization of T_1 spaces.

Lemma 11. *A space X is T_1 if and only if for every $x \in X$, every space Y, every non-closed subspace $A \subset Y$, and every point $y \in \overline{A} \backslash A$, the constant map $A \to x$ admits a unique continuous extension to $A \cup \{y\}$.*

Proof. Suppose X is not T_1. Let $Y = [0, 1]$, $A = [0, 1)$, and $y = 1$. Obtain $x \in X$ and $b \in \overline{\{x\}} \backslash \{x\}$. The constant map $A \to x$ admits two continuous extensions (mapping $1 \to x$ and $1 \to b$).

Conversely, suppose X is T_1, Y is a space, $A \subset Y$, and $y \in \overline{A} \backslash A$. If $x \in X$, the constant map $A \cup \{y\} \to x$ extends the map $A \to x$. Define $f : A \cup \{y\} \to X$ such that $f(A) = x$ and $f(y) = b$ where $b \neq x$. Since X is T_1 obtain $U \subset X$ such that U is open in X, $b \in U$, and $x \notin U$. Since $f^{-1}(U) = \{y\}$ and $\{y\}$ is not open in $A \cup \{y\}$, f is not continuous. $\square$

Theorem 12. *A space X is a J-convergence space if and only if for each non-closed $A \subset X$ at least one of the following two conditions hold:*

1. *There exists an almost closed subspace $I \subset J \cup \{\infty_J\}$ and a continuous injection $h : I \cup \{\infty_I\} \to X$ such that $h(I) \subset A$ and $h(\infty_I) \notin A$*

2. *There exists $x \in A$ such that $\{x\} \neq \overline{\{x\}}$.*

Proof. Suppose A is a non-closed subset of X.

For the first direction, suppose X is a J-convergence space. Obtain $I \subset J$ such that I is almost closed in $I \cup \{\infty_I\}$ and a map $f : I \cup \{\infty_I\} \to X$ such that $f(I) \subset A$ and $f(\infty_I) \notin A$.

Case 1. Suppose there exists $i \in I$ such $B_i = f^{-1}(f(i))$ is unbounded in I. Then $\{\infty_I\}$ is a limit point of B_i and $f|(B_i \cup \{\infty_I\})$ is continuous. Thus if $x = f(i)$ and $y = f(\infty_I)$ then $y \neq x$ (since $x \in A$ and $y \notin A$) and $y \in \overline{\{x\}}$.

Case 2. Suppose $f^{-1}(f(i))$ is bounded for all $i \in I$. Then condition 1. follows from Lemma 10.

To prove the converse, suppose at least one of conditions 1. and 2. hold. If condition 1. holds, obtain h as in condition 1. and let $f = h$. If condition 2. holds, obtain $x \in A$ and $y \neq x$ such that $y \in \overline{\{x\}}$. Let $f(J) = x$ and $f(\infty_J) = y$ and note that f is continuous by Lemma 11. $\square$

3.2 On the category of J-convergence spaces

Fixing a non-compact, well-ordered space J, let $\mathbf{TOP}_J$ denote the full subcategory of $\mathbf{TOP}$ (the usual category of topological spaces) consisting of J-convergence spaces. Let $\mathscr{J}$ be the set of compact, well-ordered spaces $I \cup \{\infty_I\}$ where I is almost closed in $J \cup \{\infty_J\}$. According to Proposition 8, a J-convergence space is precisely one which has the final topology with respect to the set of all continuous functions from elements of $\mathscr{J}$. Therefore $\mathbf{TOP}_J$ is the coreflective hull of $\mathscr{J}$, i.e. every J-convergence space is the quotient of a topological sum of elements of $\mathscr{J}$ [1].

Remark 13. If C is a closed, non-empty subspace of $J \cup \{\infty_J\}$, then there is a continuous retraction $r : J \cup \{\infty_J\} \to C$ (See [4, Proposition 2.5] for a more general "folklore" result). In particular, if K is almost closed in $J \cup \{\infty_J\}$, then $K \cup \{\infty_K\}$ is a quotient of $J \cup \{\infty_J\}$. Therefore the category $\mathbf{TOP}_J$ can be characterized as the coreflective hull of the singleton $\{J \cup \{\infty_J\}\}$. This fact does not simplify the proof of Theorem 23; we find it more convenient to use the generating set $\mathscr{J}$.

Remark 14. By [1, Lemma 0.2] every J-convergence space is the quotient of a topological sum of elements of $\mathscr{J}$. Consequently the category $\mathbf{TOP}_J$ is closed under topological sums, and it follows from Definition 6 that $\mathbf{TOP}_J$ is also closed under forming quotient spaces, and taking closed subspaces.

Lemma 15. *If Y is a J-convergence space and $f : Y \to X$ is continuous, then $f : Y \to \mathbf{J}X$ is continuous. Consequently, $\mathbf{TOP}_J$ is the coreflective hull of $\mathscr{J}$; the functor $\mathbf{J} : \mathbf{TOP} \to \mathbf{TOP}_J$ is the coreflection.*

Remark 16. It is well-known that the sequential category is not closed under direct products (with the usual product topology) [10, Example 1.11]. It is no surprise the same difficulty arises in the context of J-convergence spaces: if X and Y are J-convergence spaces, then $X \times Y$ is not necessarily a J-convergence space. In fact, there is a sequential space G such that $G \times G$ is not a J-convergence space for any J (See Example 27).

Since $\mathbf{TOP}_J$ is a coreflective hull of a collection of compact Hausdorff spaces, we are motivated to use Theorem 4.4 of [1] to show that $\mathbf{TOP}_J$ inherits the structure of a Cartesian closed category (which is, in fact, a convenient category of topological spaces, in the sense of [25], since it contains the sequential category [1, Proposition 7.3]). We follow the usual construction of an internal product and function space for coreflective hulls [1].

Definition 17. The **J-product** of J-convergence spaces X and Y is $X \times_J Y = \mathbf{J}(X \times Y)$.

The J-product clearly defines a categorical product since $\mathbf{TOP}_J$ is coreflective. Functions spaces are given the so-called $\mathscr{J}$-open topology [1, Definition 0.1].

Definition 18. Given J-convergence spaces X, Y, the set of all continuous functions $X \to Y$ is denoted $M(X,Y)$. For $K \cup \{\infty_K\} \in \mathscr{J}$, a map $t : K \cup \{\infty_K\} \to X$, and an open set $U \subseteq Y$, let $W(t,U) = \{f \in M(X,Y) | ft(K \cup \{\infty_K\}) \subseteq U\}$. The **$\mathscr{J}$-open topology** on $M(X,Y)$ is the topology generated by the subbasic sets $W(t,U)$ for all $K \cup \{\infty_K\} \in \mathscr{J}$, maps $t : K \cup \{\infty_K\} \to X$ and open $U \subseteq Y$. Let $M_J(X,Y)$ denote $M(X,Y)$ with the $\mathscr{J}$-open topology.

It is not necessarily true that $M_J(X,Y)$ is a J-convergence space. Therefore, we take the function space in $\mathbf{TOP}_J$ to be the coreflection $\mathbf{J}M_J(X,Y)$.

To provide context for our proof of the Cartesian closedness of $\mathbf{TOP}_J$ (See Remark 20), we give the following definition.

Definition 19. Recall X is a Frechet Urysohn space if for each subspace $A \subset X$ and each limit point $x \in A$, there exists a sequence $\{a_n\} \subset A$ such that $a_n \to x$. Define X to be a **J-Frechet Urysohn space** if the following two statements are equivalent for all subsets $A \subset X$. 1. A is not a closed subspace of X. 2. For all points $x \in \overline{A} \backslash A$, the subspace $A \cup \{x\}$ is a J-convergence space.

Every Frechet Urysohn space is sequential; the proof generalizes immediately to give that every J-Frechet Urysohn space is a J-convergence space.

Remark 20. Our proof of the Cartesian closedness of $\mathbf{TOP}_J$ (See Theorem 23) requires us to show the direct product $(J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$ is a J-convergence space. While not in itself a deep result, to see why this not totally obvious recall the following obstructions to a simple proof: 1. On the one hand the space $J \cup \{\infty_J\}$ is a J convergence space. However, in general, if X is a J-convergence space then $X \times X$ need not be a J-convergence space. 2. Every J-Frechet Urysohn space is a J-convergence space and the compact, well-ordered space $J \cup \{\infty_J\}$ is a J-Frechet Urysohn space. However $(J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$ is not a J-Frechet Urysohn space [30], and thus we cannot immediately deduce that $(J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$ is a J-convergence space.

The following Lemma carries the main idea for the subsequent proof that $(J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$ is a J-convergence space.

Lemma 21. *Suppose K is a compact, well-ordered space with minimal element 0. Suppose $(M,m) \in K \times K$. Suppose $B \subset [0,M] \times [0,m]$ such that for all $k < M$, $([0,k] \times [0,m]) \cap B$ is closed in $[0,M] \times [0,m]$ and such that $\emptyset = (\{M\} \times [0,m]) \cap B = ([0,M] \times \{m\}) \cap B$. Suppose $\{(M,m)\} \in \overline{B} \backslash B$. Then there exists a limit point $l \in [0,M]$ and a map $f : [0,l] \to [0,M] \times [0,m]$ such that $f([0,l)) \subset B$ and $f(l) = (M,m)$.*

Proof. We will define $f : [0,l] \to [0,M] \times [0,m]$ so that if $f(k) = (x(k), y(k))$ then each of the maps x and y is strictly increasing.

To achieve this, at each stage of the definition of f we make as little strict progress in the direction of $[0,M]$ as possible, while guaranteeing positive progress in the direction of $[0,m]$. Thus, to implement the transfinite recursive definition of f, if k is not a limit point of $[0,M]$, (and working within B) $f(k)$ is defined so that "starting at $f(k-1)$ we move our current abscissa as little as possible strictly to the right subject to the demand that strict vertical progress is possible at the new abscissa. Then, having selected our new abscissa, we then claim as much vertical progress as possible. If k is a limit point of $[0,M]$ then continuity of $f|_{[0,k)}$ (and compactness of $[0,M] \times [0,m]$) forces the definition of $f(k)$ to be the unique value such that $f|_{[0,k]}$ is continuous.

Before defining f, we build a few basic observations following directly from our hypotheses and previous observations.

Observation 0: For all $k < M$, $(\{k\} \times [0,m]) \cap B$ is closed in $[0,M] \times [0,m]$.

Observation 1: M is a limit point of $[0,M]$. (To obtain a contradiction, suppose otherwise. Then $\{M\}$ is open in $[0,M]$, and hence, since (M,m) is a limit point of B, the open set $\{M\} \times [0,m]$ contains a point $(M, y_m) \in B$ such that $(M, y_m) \neq (M,m)$, contradicting the hypothesis that $\emptyset = (\{M\} \times [0,m]) \cap B$).

Observation 2: By a symmetric argument applied to Observation 1, m is a limit point of $[0,m]$.

Observation 3: By Observations 1 and 2, basic open sets $U \times V$ of $[0,M] \times [0,m]$ containing (M,m) are of the form $(a,M] \times (b,M]$ with $a < M$ and $b < M$.

Observation 4: If W is an open set of $[0,M] \times [0,m]$ such that $(M,m) \in W$ then (since (M,m) is a limit point of B) Observation 3 ensures there exists $(x,y) \in W \cap B$ such that $x < M$ and $y < m$ and also, for each $(x,y) \in W \cap B$ there exists $(x^*, y^*) \in W \cap B$ such that $x < x^* < M$ and $y < y^* < m$.

For each $k \in [0,M]$ define $B_k = (\{k\} \times [0,m]) \cap B$. If $B_k \neq \emptyset$ (by Observation 0) let m_k be minimal such that $B_k \subset \{k\} \times [0, m_k]$ (Thus (k, m_k) is the 'maximal' element of B_k).

Note $B \neq \emptyset$ since $\overline{B} \neq \emptyset$. Obtain $x_0 \in [0,M]$ minimal such that $B_{x_0} \neq \emptyset$. Define $f(0) = (x_0, m_0)$ and let $y_0 = m_0$. By hypothesis $x_0 < M$ and $y_0 < M$.

Suppose $k \in [0,M]$ and $f(i) \in B$ for all $i < k$ so that all of the following hold:

i) If $i < k$ then $f(i) = (x_i, y_i) \in B$.

ii) If $i < k$ then $i \leq y_i$ and $i \leq x_i$.

iii) If $i < j < k$ then $x_i < x_j < M$ and $y_i < y_j < m$.

iv) $f|_{[0,k)}$ is continuous.

If $k-1$ exists obtain $x_k \in [0, M]$ minimal such that $x_{k-1} < x_k < M$, $B_k \neq \emptyset$, and $y_{k-1} < m_k < m$. Define $f(k) = (x_k, m_k)$ and let $y_k = m_k$.

To see that $f(k)$ is well defined let $W = (x_{k-1}, M] \times (y_{k-1}, M]$. Observation 4 ensures the existence of the desired $f(k)$. Conditions i) and iii) are preserved by definition. To check condition ii), let $i = k-1$. Thus $k-1 \leq x_{k-1}$ and $k-1 \leq y_{k-1}$. Thus $(k-1)+1 \leq x_k$ and $(k-1)+1 \leq y_k$. For condition iv), notice $[0, k+1) = [0, k) \cup \{k\}$ and $[0, k) = [0, k-1]$. Thus $[0, k-1)$ is the union of two disjoint closed sets and hence continuity of $f|_{[0,k+1)}$ follows from the familiar pasting from general topology.

If $k-1$ does not exist then k is a limit point of K, and define $f(k) = (\sup_{i<k}\{x_i\}, \sup_{i<k}\{y_i\}) = (x_k, y_k)$. Condition iii) and the l.u.b. property of K ensure $f(k)$ is well defined.

To check continuity of $f|_{[0,k]}$ suppose W is a basic open set of $[0, M] \times [0, m]$. Let $U = f|_{[0,k]}^{-1}(W)$. To check iv) If $(M, m) \notin W$ then continuity of $f|_{[0,k)}$ ensures U is open in $[0, k)$ and, (since $[0, k)$ is open in $[0, k]$), U is open in $[0, k]$. Suppose $(M, m) \in W$. Then (since $f|_{[0,k)}$ is increasing) there exists $a \in [0, M)$ such that $f_{[0,k)}^{-1}(W) = (a, k)$. Thus $f_{[0,k]}^{-1}(W) = (a, k) \cup \{k\} = (a, k]$. Since $(a, k]$ is open in $[0, k]$, we conclude $f|_{[0,k]}$ is continuous. By definition $f|_{[0,k+1)} = f|_{[0,k]}$ and thus condition iv) is preserved since $f|_{[0,k+1)}$ is continuous.

Condition ii) for $i < k$ combined with continuity of $f|_{[0,k]}$ ensure condition ii) is preserved for $i \leq k+1$.

By definition $f(k) = (x_k, y_k)$. Preservation of the remaining conditions depend on whether $f(k) \in B$ or not.

Case 1. If $f(k) \in B$ then by hypothesis of the Lemma, $x_i < M$ and $y_i < M$. Moreover, since $\{x_i\}$ and $\{y_i\}$ for $i \leq k$ are strictly transfinite sequences, condition iii) is preserved.

Case 2. Suppose $f(k) \notin B$. Then, by continuity of f, $f(k)$ is a limit point of B.

Observation 5 ensures (M, m) is the only limit point of B, and hence $f(k) = (M, m)$. Let $l = k$ and the Lemma at hand is proved. □

Lemma 22. *$(J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$ is a J-convergence space.*

Proof. Suppose $A \subset (J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$ and A is not closed.

We seek I almost closed in $J \cup \{\infty_J\}$ and a map $f : I \cup \{\infty_I\} \to (J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$ such that $f(I) \subset (J \cup \{\infty_J\})$ and $f(\infty_I) \notin (J \cup \{\infty_J\})$.

First we reduce as follows to the case that each "vertical or horizontal slice" of A is closed in $(J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$. For each $x \in J \cup \{\infty_J\}$ define $B_x = A \cap (\{x\} \times (J \cup \{\infty_J\}))$. Note each subspace $\{x\} \times (J \cup \{\infty_J\})$ is closed in $(J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$ and is also canonically homeomorphic to the J-convergence space $J \cup \{\infty_J\}$. Thus, if there exists $x \in J \cup \{\infty_J\}$ such that B_x is not closed in $(J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$, then there exists I almost closed $J \cup \{\infty_J\}$ and there exists a map $f : I \cup \{\infty_I\} \to \{x\} \times (J \cup \{\infty_J\})$ such that $f(I) \subset B_x$ and $f(\infty_I) \notin B_x$. Since $\{x\} \times (J \cup \{\infty_J\})$ is closed in $(J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$, it follows that $f(\infty_I) \in \{x\} \times (J \cup \{\infty_J\})$. Hence $f(\infty_I) \notin A$ and we have the desired map f.

After applying a symmetric argument to slices of the form $(J \cup \{\infty_J\}) \times \{y\}$ we have reduced to the case that the subspaces $(\{x\} \times (J \cup \{\infty_J\})) \cap A$ and $(J \cup \{\infty_J\}) \times \{y\}) \cap A$ are closed in $(J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$ for all $\{x, y\} \subset (J \cup \{\infty_J\})$.

Note $(\{0\} \times K) \cap A$ is closed and $(K \times K) \cap A$ is not closed. Hence there exists $M \in K$ minimal such that $([0, M] \times K) \cap A$ is not closed. By our assumptions $([0, M] \times \{0\}) \cap A$ is closed. Thus there exists $m \in K$ minimal such that $([0, M] \times [0, m]) \cap A$ is not closed in $K \times K$.

Let $C = ([0, M] \times [0, m]) \cap A$. Since $[0, M] \times [0, m]$ is closed in $K \times K$, C is not closed in $[0, M] \times [0, m]$. Hence $\overline{C} \backslash C \neq \emptyset$ and $\overline{C} \backslash C \subset [0, M] \times [0, m]$. Suppose $(x, y) \in \overline{C} \backslash C$. To obtain a contradiction suppose $x < M$. Then (x, y) is a limit point of the closed set $([0, x] \times [0, m]) \cap A$ and hence $(x, y) \in A$, a contradiction. Thus $x = M$. Suppose, to obtain a contradiction $y < m$. Then (M, y) is a limit point of the closed set $([0, M] \times [0, y]) \cap A$ and hence $(M, y) \in A$, a contradiction. Hence $\overline{C} \backslash C = \{(M, m)\}$.

Note (M, m) is not in the closed set $(([0, M] \times \{m\}) \cup (\{M\} \times [0, m])) \cap C$. Obtain $a \in [0, M)$ and $b \in [0, m)$ such that $([a + 1, M] \times \{m\}) \cap C = (\{M\} \times [b + 1, m]) \cap C = \emptyset$. Note $[a + 1, M] \times [b + 1, m]$ is clopen in $[0, M] \times [0, m]$. Let $B = [a + 1, M] \times [b + 1, m] \cap C$. Note (K, B, M, m) satisfy the hypothesis of Lemma 21 and obtain a limit point $l \in [0, M]$ and a map $f : [0, l] \to \overline{B}$ such that $f([0, l)) \subset B \subset A$ and $f(l) \notin B$. Let $I = [0, l)$, note $[0, l)$ is almost closed in J, and $\infty_I = l$. By definition $\overline{B} \backslash B \subset \overline{C} \backslash C \subset \overline{A} \backslash A$ and hence $f(l) \in A$. $\square$

Theorem 23. *The category* $\mathbf{TOP}_J$ *with* J*-product* $X \times_J Y$ *and function space* $\mathbf{J}M_J(X, Y)$ *is Cartesian closed, i.e. for any* J*-convergence spaces* X, Y, Z*, there is a natural homeomorphism*

$$\mathbf{J}M_J(X, \mathbf{J}M_J(Y, Z)) \cong \mathbf{J}(X \times_J Y, Z).$$

Proof. The theorem follows directly from Theorem 4.4 of [1] once the following two conditions are verified: 1. for each $A, B \in \mathscr{J}$, the direct product $A \times B$ is a J-convergence space and 2. $\mathscr{J}$ is a regular class of spaces [1, Definition 2.2] ($\mathscr{J}$ is regular if for each element $k \in K \cup \{\infty_K\}$ with $K \cup \{\infty_K\} \in \mathscr{J}$, every neighborhood U of k in $K \cup \{\infty_K\}$ contains a closed neighborhood C for which there is a surjection $s : B \to C$ with $B \in \mathscr{J}$.

1. If $K \cup \{\infty_K\}, L \cup \{\infty_L\} \in \mathscr{J}$, then $(K \cup \{\infty_K\}) \times (L \cup \{\infty_L\})$ is a compact and thus closed subset of $(J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$. Since $(J \cup \{\infty_J\}) \times (J \cup \{\infty_J\})$ is a J-convergence space (Lemma 22) and $\mathbf{TOP}_J$ is closed under taking closed subsets (Remark 14), $(K \cup \{\infty_K\}) \times (L \cup \{\infty_L\})$ is a J-convergence space.

2. If K is almost closed in $J \cup \{\infty_J\}$ and k is an isolated point of $K \cup \{\infty_K\}$, the constant map $J \cup \{\infty_J\} \to K \cup \{\infty_K\}$ at k suffices. If k is a limit point, we may assume $U = (k_0, k]$ for $k_0 < k$. There is a point k' such that $k_0 < k' < k$. Note the interval $L = [k', k)$ in K is almost closed in $K \cup \{\infty_K\}$ and is therefore almost closed in $J \cup \{\infty_J\}$ (See Corollary 5). Thus $C = [k', k] = L \cup \{\infty_L\} \in \mathscr{J}$ and the identity $C \to C$ is obviously surjective. $\square$

An immediate application of Theorem 23 is the characterization of quotient maps whose direct product is again quotient. In general, if $q : X \to Y$ is a quotient map, then $q \times q : X \times X \to Y \times Y$ can fail to be a quotient map; confirming that $q \times q$ is quotient is highly sensitive to the context [5][21], and category [1][25].

Lemma 24. *Let X be a J-convergence space and $q : X \to Y$ be a quotient map. For any space Z, $q \times_J id : X \times_J Z \to Y \times_J Z$ is a quotient map.*

Proof. Certainly, $q \times id$ is continuous. Using Lemma 15, it suffices to verify the universal property with respect to maps $X \times_J Z \to W$ to a J-convergence space W. Suppose W is a J-convergence space and $f : X \times_J Z \to W$ is a map such that $f(x_1, y) = f(x_2, y)$ whenever $q(x_1) = q(x_2)$. There is a unique function $g : Y \times Z \to W$ such that $g \circ (q \times id) = f$; it suffices to show $g : Y \times_J Z \to W$ is continuous. The adjoint of f is a map $f' : X \to \mathbf{J}M_J(X, W)$. Note $f'(x_1)(z) = f(x_1, z) = f(x_2, z) = f'(x_2)(z)$ whenever $q(x_1) = q(x_2)$. Therefore there is a unique continuous map $g' : Y \to \mathbf{J}M_J(Z, W)$ such that $g' \circ q = f'$. The adjoint of g', which is continuous, is precisely $g : Y \times_J Z \to W$ since

$$g \circ (q \times id)(x, z) = g(q(x), z) = g'(q(x))(z) = f'(x)(z) = f(x, z).$$

□

Corollary 25. *If X and X' are J-convergence spaces and $q : X \to Y$ and $q' : X' \to Y'$ are quotient maps, then $q \times_J q' : X \times_J X' \to Y \times_J Y'$ is a quotient map.*

Corollary 26. *If X and X' are J-convergence spaces such that $X \times X'$ is a J-convergence space and $q : X \to Y$ and $q' : X' \to Y'$ are quotient maps, the direct product $q \times q' : X \times X' \to Y \times Y'$ is a quotient map iff $Y \times Y'$ is a J-convergence space.*

Proof. Since quotients of J-convergence spaces are J-convergence spaces one direction is obvious. If $Y \times Y'$ is a J-convergence space, then both $X \times X' = X \times_J X'$ and $Y \times Y' = Y \times_J Y'$. Thus $q \times q'$ is quotient by Corollary 25. □

Finally, we give an example of a sequential space G whose self-product $G \times G$ is not a J-convergence space for any J.

Example 27. Suppose X is a sequential space (and thus a J-convergence space for every J) whose self-product $X \times X$ is also sequential and $q : X \to G$ is a quotient map such that $q \times q : X \times X \to G \times G$ fails to be a quotient map. Since $\mathbf{TOP}_J$ is closed under quotients, G is a J-space for every J but Corollary 26 guarantees that $G \times G$ cannot be a J-space for any J. This situation is realized in [9]: $X = \Omega(HE, p)$ is taken to be the loop space of the Hawaiian earring with the compact-open topology (which is metrizable and therefore $X \times X$ is sequential) and $G = \pi_1(HE, p)$ is the fundamental group viewed as the quotient with respect to the canonical map $\Omega(HE, p) \to \pi_1(HE, p)$ identifying homotopy classes.

3.3 Comparison to compactly generated and transfinite sequential spaces.

Let $\cup\mathbf{TOP}_J$ denote the category of spaces X which are J-convergence spaces for some J, $\mathbf{CG}$ be the category of compactly generated spaces, $\mathbf{TF}$ be the category of transfinite sequential spaces and $\mathbf{CG} \cap \mathbf{TF}$ denote the category of compactly generated, transfinite sequential spaces. We show $\cup\mathbf{TOP}_J$ constitutes a strict subcategory of $\mathbf{CG} \cap \mathbf{TF}$. On the other hand, three examples show

1. $\cup\mathbf{TOP}_J \neq \mathbf{CG} \cap \mathbf{TF}$ - See Example 31
2. $\mathbf{CG} \backslash \mathbf{TF} \neq \emptyset$ - See Example 33
3. $\mathbf{TF} \backslash \mathbf{CG} \neq \emptyset$ - See Example 34

Here, set theoretic notation (intersection and complement) is used for classes of objects; since all categories are full subcategories of $\mathbf{TOP}$, no confusion should arise regarding morphisms. Transfinite sequential spaces are more general than J-convergence spaces [17] and defined as follows.

Definition 28. Suppose $J \cup \{\infty_J\}$ is a well-ordered set such that J is unbounded. By the **directed topology** on $J \cup \{\infty_J\}$ we mean the topology generated by sets $\{j\}$ and $(j, \infty_J]$ such that $j \in J$. (See also [17]) a space X is a **transfinite-sequential space** if for each non-closed set $A \subset X$, there exists an unbounded, well-ordered set J and a function $f : J \cup \{\infty_J\} \to X$ such that $f(J) \subseteq A$ and $f(\infty_J) \notin A$, and such that f is continuous with respect to the directed topology on $J \cup \{\infty_J\}$.

Lemma 29. *Every J-convergence space is a transfinite sequential space.*

Proof. Suppose X is a J-convergence space and A is not closed in X. Let $J \cup \{\infty_J\}$ denote the Alexandorff compactification of J and let $\hat{J} \cup \{\infty_J\} = J \cup \{\infty_J\}$ with the directed topology. Note $\hat{id} : \hat{J} \cup \{\infty_J\} \to J \cup \{\infty_J\}$ is continuous (since the directed topology of $\hat{J} \cup \{\infty_J\}$ is finer than the order topology of $J \cup \{\infty_J\}$). Obtain a map $f : J \cup \{\infty_J\} \to X$ such that $f(J) \subset A$ and $f(\infty_J) \notin A$. The map $\hat{f} = f \circ \hat{id} : \hat{J} \cup \{\infty_J\} \to X$ shows X is a transfinite sequential space. □

Recall the space X **compactly generated** (i.e. X is $\mathbf{CG}$) if for each non-closed $A \subset X$ there exists a compact Hausdorff space K and a map $f : K \to X$ such that $f^{-1}(A)$ is not closed in K [26].

In the language of category theory, the space X is compactly generated if it is in the coreflective hull of the class $\mathscr{CH}$ of compact Hausdorff spaces, or equivalently, if it has the final topology with respect to all continuous functions $K \to X$ where K is compact Hausdorff. The compactly generated category $\mathbf{CG}$ is Cartesian closed and is frequently used as a "convenient" category in which to do algebraic topology [25][26]. Let $\mathbf{k} : \mathbf{TOP} \to \mathbf{CG}$ denote the coreflection: For any space X, $\mathbf{k}X$ has the topology consisting of sets U such that $f^{-1}(U)$ is open in K for every $K \in \mathscr{CH}$ and map $f : K \to X$. Additionally, let $X \times_k Y = \mathbf{k}(X \times Y)$ denote the usual product of compactly generated spaces.

Lemma 30. *Every J-convergence space is compactly generated. Consequently, for every space X, the identity function $\mathbf{J}X \to \mathbf{k}X$ is continuous.*

Proof. Since $\mathscr{J} \subset \mathscr{CH}$, the coreflective hull of $\mathscr{J}$ is a subcategory of the coreflective hull of $\mathscr{CH}$. Thus $\mathbf{TOP}_J \subset \mathbf{CG}$.

Since $\mathbf{J}X$ is $\mathbf{CG}$, applying the functor $\mathbf{k}$ to the continuous identity function $\mathbf{J}X \to X$ yields $\mathbf{J}X = \mathbf{kJ}X \to \mathbf{k}X$. □

The inclusion of categories $\cup\mathbf{TOP}_J \subset \mathbf{CG} \cap \mathbf{TF}$ follows; the following example shows this inclusion is strict.

Example 31. *Let X be an uncountable, well-ordered set with the discrete topology and let $Y = X \cup \{\infty\}$ denote the Alexandroff compactification of X. Then Y is a compact Hausdorff space, and Y is a transfinite sequential space, but for Y is not a J-convergence space for any non-compact, well-ordered space J.*

Proof. Since Y is a compact Hausdorff space, A is not closed in Y iff A is not compact. Thus $id : Y \to Y$ shows Y is compactly generated.

To see that Y is transfinite sequential, suppose $A \subset Y$ and A is not closed. Then ∞ is a limit point of A. Notice $A \cup \infty$ is a well-ordered set. Let $A\hat{} \cup \infty\hat{} = A \cup \infty$ with the directed topology and consider the inclusion function $j : A\hat{} \cup \infty\hat{} \to Y \cup \infty$. If $U \subset Y$ is an open set such that $\infty \in U$, then $Y \backslash U$ is finite. Select $y \in Y$ such that $Y \backslash U < y$. Hence if $y < a$ and $a \in A$ then $j(a) \in U$. Thus Y is transfinite sequential.

Suppose J is a non-compact, well-ordered space with one point compactification $J \cup \{\infty_J\}$. To see that Y is not a J space, suppose I is almost closed in $J \cup \{\infty_J\}$ and suppose $f : I \cup \{\infty_I\} \to Y$ is a map such that $f(I) \subset A$. Note if $i \in I$ then $f([0, i])$ is a compact subset of the discrete space A, and hence $f([0, i])$ is finite.

To obtain a contradiction suppose for each integer $N \geq 1$ there exists $i_N \in I$ such that $|f([0, i_N])| = N$. Then $i_1 < i_2$......

Since X is uncountable there exists a limit $i \in I$. Then $|f([0, i])| > N$ for each N. However $|f([0, i])|$ is finite and we have a contradiction. Since f is continuous at ∞_I we conclude $f(\infty_I) \in f(I)$ and hence Y is not a J-convergence space. □

Examples show the categories of compactly generated spaces and transfinite sequential spaces are unrelated.

Lemma 32. *Suppose X is a Hausdorff space and there exists a countably infinite set $A = \{a_1, a_2, ..\} \subset X$ such that A is not closed, and such that each convergent sequence in A is eventually constant. Then X is not a transfinite sequential space.*

Proof. Let J be an unbounded, well-ordered set and $J \cup \{\infty_J\}$ has the directed topology as in Definition 28. To obtain a contradiction suppose $f : J \cup \{\infty_J\} \to X$ is a map such that $f(J) \subset A$ and $f(\infty_J) \notin A$. Let $S_n = f^{-1}(a_n)$ and let $k_n = \sup S_n$. Note the singleton $f(S_n) \subset A$. Hence, since X is a T_2 space,

$f(\overline{S_n}) = f(S_n)$. Thus $k_n < \infty_J$ for all n (since otherwise $f(\infty_J) \in A$). Let $s_n = \max\{k_1, .., k_n\}$. Note $s_1 \leq s_2....$ and $s_n < \infty_J$. Let $s = \sup\{s_n\}$ and note $s = \infty_J$ (since otherwise we obtain the contradiction $f((s, \infty_J))) \cap A = \emptyset$). By construction, the sequence $s_n \to \infty_J$. By continuity of f, $f(s_n) \to f(\infty_J)$. Thus $\{f(s_n)\}$ is a convergent sequence in A and hence $\{f(s_n)\}$ is eventually constant. Thus $f(\infty_J) \in A$ and we have a contradiction. $\square$

Example 33. ($\mathbf{CG} \backslash \mathbf{TF} \neq \emptyset$) Let ω denote the natural numbers with the discrete topology and let $X = \beta\omega$, the Stone-Čech compactification of ω. (See [8]). Then X is a compactly generated space but is not a transfinite sequential space.

Proof. Since $\beta\omega$ is compact Hausdorff, it is compactly generated. Let $A = \omega \subset X$. By construction, ω is not closed in $\beta\omega$. Each bounded real valued function $f : \omega \to \mathbb{R}$ is the restriction of a map $f : \beta\omega \to \mathbb{R}$. Thus each convergent sequence in ω is eventually constant. Otherwise, there exists a convergent subsequence $n_1 < n_2.....$ with $n_i \to x \in X$. Since X is Hausdorff, the bounded real valued map such that $f(n_{2i}) = 0$ and $f(m) = 1$ if $m \neq n_{2i}$ cannot be continuously extended to $\{x, \omega\}$ and we have a contradiction. Now apply Lemma 32 to the data (X, A). $\square$

Example 34. ($\mathbf{TF} \backslash \mathbf{CG} \neq \emptyset$) Suppose J is an uncountable, unbounded, well-ordered set and $X = J \cup \{\infty_J\}$ with the directed topology. Then X is a transfinite sequential space but is not compactly generated.

Proof. Suppose $A \subset X$ and A is not closed in X. Then $\overline{A} \backslash A \neq \emptyset$ and hence $\{\infty_J\} = \overline{A} \backslash A$ since ∞_J is the only limit point of X.

To see that $X \in \mathbf{TF}$ let $Y = A \cup \{\infty_J\}$ with the subspace topology. Note Y is a well-ordered set, A is unbounded in Y, and Y enjoys the directed topology. Thus inclusion $id : Y \to X$ is continuous, $id(A) \subset A$ and $id(\infty_J) \notin A$.

To see that $X \notin \mathbf{CG}$, note J is not closed in X. Suppose K is a compact T_2 space and $f : K \to X$ is a map. Let $B = f^{-1}(J)$. It suffices to prove B is closed in K. Note $f(K)$ is compact. To obtain a contradiction suppose $f(K)$ is infinite. Obtain j_1 minimal in $f(K)$ and note $j_1 \in J$. Suppose $n > 1$ and $j_1 < j_2... < j_{n-1}$ have been selected such that $j_i \in J \cap f(K)$. Obtain $j_n \in f(K)$ minimal such that $n_{n-1} < n_n$. Note $j_n \in J$. Let $C = \{j_1, j_2, ...\}$. Since J is uncountable there exists $l \in J$ such that $C < j$. Note $f(K)$ is Hausdorff since X is Hausdorff, and note C is a closed subspace of X. Hence C is a closed subspace of $f(K)$ and thus C is compact. However C is an infinite space with the discrete topology, and the thus C is not compact, and we have a contradiction. Thus $f(K)$ is a finite set. Let $f(K) \backslash \{\infty_J\} = \{k_1, k_2, .., k_n\}$. Since X is Hausdorff, $\{k_i\}$ is closed for each i. Thus, since f is continuous, B is the union of finitely many closed sets $f^{-1}\{k_i\}$. Hence $f^{-1}(J)$ is closed. Thus $X \notin \mathbf{CG}$. $\square$

Despite the fact that $\mathbf{TOP}_J$ is a proper subcategory of $\mathbf{CG}$, we find in Theorem 36 below that the topologies on the topologies of $X \times_J Y$ and $X \times_k Y$ agree when X and Y are J-convergence spaces.

Lemma 35. *[26] If X and X' are compactly generated and $q : X \to Y$ and $q' : X' \to Y'$ are quotient maps, then $q \times_k q' : X \times_k X' \to Y \times_k Y'$ is a quotient map.*

Theorem 36. *If X and Y are J-convergence spaces, then $X \times_k Y = X \times_J Y$.*

Proof. Since every J-convergence space is compactly generated, the identity functions $X \times_J Y \to X \times_k Y \to X \times Y$ are continuous (Lemma 30). Applying the functor $\mathbf{J}$ gives $X \times_J Y \to \mathbf{J}(X \times_k Y) \to X \times_J Y$ and thus $\mathbf{J}(X \times_k Y) = X \times_J Y$. Thus it suffices to show $X \times_k Y$ is a J-convergence space.

Since $\mathbf{TOP}_J$ is the coreflective hull of $\mathscr{J}$, every J-convergence space is the quotient of a topological sum of spaces of the form $K \cup \{\infty_K\}$ where K is almost closed in $J \cup \{\infty_J\}$. Let W and Z be such topological sums for X and Y respectively and let $q : W \to X$ and $r : Z \to Y$ be the quotient maps. Now the direct product $q \times_k r : W \times_k Z \to X \times_k Y$ is a quotient map by Lemma 35. Note the direct product $W \times Z$ is a topological sum of products $(K \cup \{\infty_K\}) \times (L \cup \{\infty_L\})$ where K and L are almost closed in $J \cup \{\infty_J\}$. All such products are closed subspaces of $J \cup \{\infty_J\} \times J \cup \{\infty_J\}$ (which is a J-convergence space by Lemma 22) and are therefore J-convergence spaces. Since $\mathbf{TOP}_J$ is closed under sums, $W \times Z$ is a J-convergence space. By Lemma 30, $W \times Z$ is compactly generated and we have $W \times Z = W \times_k Z$. Thus $W \times_k Z$ is a J-convergence space. Finally, since $q \times_k r : W \times Z \to X \times_k Y$ is quotient and $\mathbf{TOP}_J$ is closed under forming quotients, $X \times_k Y$ is a J-convergence space. □

4 J-unique convergence spaces

Definition 37. Suppose J is a non-compact, well-ordered space. The space X is a **J-unique convergence space** (or **J-unique**), if for every almost closed subspace $I \subset J \cup \{\infty_J\}$ and every pair of maps $f, g : I \cup \{\infty_I\} \to X$ such that $f|_I = g|_I$, then $f = g$.

Lemma 38. *Every J-unique space is T_1.*

Proof. To prove the contrapositive suppose X is not T_1. Obtain $x \in X$ and $y \in \overline{\{x\}} \backslash \{x\}$. Let $f : J \cup \{\infty_J\} \to X$ be the constant map at x and define $g : J \cup \{\infty_J\} \to X$ such that $G(j) = x$ if $j \in J$ and $G(\infty_J) = y$. By constriction, both f and G are continuous, $f \neq g$ and $f|_J = g|_J$. Thus X is not J-unique. □

The following Lemma characterizes J-unique spaces in terms of continuous injections of compact subspaces of J.

Lemma 39. *A space X is J-unique if and only if the following two conditions hold:*

1. *X is T_1*
2. *For every almost closed subspace $I \subset J \cup \{\infty_J\}$, if $f, g : I \cup \{\infty_I\} \to X$ are continuous injections such that $f|_I = g|_I$, then $f = g$.*

Proof. If X is J-unique then conditions 1 and 2 follow from Lemma 38 and the definition of J-unique space.

Conversely suppose conditions 1 and 2 hold. Suppose I is almost closed in $J \cup \{\infty_J\}$, and $f, g : I \cup \{\infty_I\} \to X$ are maps satisfying $f|_I = g|_I$. We must show $f(\infty_I) = g(\infty_I)$.

Case 1. There exists $i \in I$ such that $A_i = f^{-1}(f(i))$ is unbounded in I. Thus ∞_I is a limit point of A_i. Since X is T_1, Lemma 11 ensures the constant map $f|_{A_i}$ admits a unique extension to $A_i \cup \{\infty_I\}$. Since $g|_{A_i} = f|_{A_i}$ we conclude $g(\infty_I) = f(\infty_I)$.

Case 2. If $f^{-1}(f(i))$ is bounded in I for each i, then, by Lemma 10, there exists a closed, cofinal subspace $K' \subset I$ such that $f|_{K'}$ is one to one. Note that K' is almost closed in $J \cup \{\infty_J\}$ and $\infty_{K'} = \infty_I$. If $f(\infty_I)$ and $g(\infty_I)$ do not lie in $f(K')$, let $K = K'$. On the other hand, if $f^{-1}(f(\infty_I)) \cup g^{-1}(g(\infty_I)) \neq \emptyset$, take k be minimal in K' such that $k > f^{-1}(f(\infty_I)) \cup g^{-1}(g(\infty_I))$ and let $K = K' \backslash [0, k]$. Note that K is closed and cofinal in K' (and thus almost closed in $J \cup \{\infty_J\}$) and $\infty_K = \infty_{K'}$. Since $f|_K, g|_K : I \cup \{\infty_I\} \to X$ are continuous injections such that $f|_K = g|_K$, condition 2. gives $f(\infty_I) = f(\infty_K) = g(\infty_K) = g(\infty_I)$. ☐

Recall, the space X is a **KC-space** if every compact subspace of X is closed in X.

Corollary 40. *Every KC-space is J-unique.*

Proof. Recall X is a KC space if every compact subset of X is closed in X. Since X is KC space X is T_1. We apply Lemma 39 to check that X is J-unique. Suppose I is almost closed in $J \cup \{\infty_J\}$ and $f, g : I \cup \{\infty_I\} \to X$ are continuous injections such that $f|_I = g|_I$. To obtain a contradiction, suppose $f(\infty_I) \neq g(\infty_I)$. Since g is an injection, $g(\infty_I) \notin f(I \cup \{\infty_I\})$. Additionally, since X is a KC space, $f(I \cup \{\infty_I\})$ is closed in X. Hence $U = X \backslash f(I \cup \{\infty_I\})$ is an open neighborhood of $g(\infty_I)$ in X. Thus $g^{-1}(U) = \{\infty_I\}$ is open in $I \cup \{\infty_I\}$. This contradicts the fact that ∞_I is a limit point of I in $I \cup \{\infty_I\}$. ☐

Lemma 41. *Suppose J is a non-compact, well-ordered space and X is a J-unique, J-convergence space. Suppose that whenever $C \subset J$ is compact and $h : C \to X$ is a continuous injection, $h(C)$ is closed in X. If $I \subset J$ is almost closed in $J \cup \{\infty_J\}$ and $h : I \cup \{\infty_I\} \to X$ is a continuous injection, then $h(I \cup \{\infty_I\})$ is closed in X.*

Proof. Since X is J-unique Lemma 38 ensures X is T_1. Suppose, to obtain a contradiction, that $h(I \cup \{\infty_I\})$ is not closed in X.

Since X is a T_1, J-convergence space, obtain, by Theorem 12, a subspace $K' \subset J \cup \{\infty_J\}$ such that K' is almost closed in $J \cup \{\infty_J\}$ and obtain a continuous injection $g' : K' \cup \{\infty_{K'}\} \to X$ such that $g'(K') \subset h(I \cup \{\infty_I\})$ and $g'(\infty_{K'}) \notin h(I \cup \{\infty_I\})$.

If $h(\infty_I) \notin g'(K' \cup \{\infty_{K'}\})$, let $K = K'$. If $h(\infty_I) \in g'(K' \cup \{\infty_{K'}\})$, let $m = (g')^{-1}(h(\infty_I))$. Note that $m \in K'$ since $g'(\infty_{K'}) \notin h(I \cup \{\infty_I\})$. Now let $K = K' \backslash [0, m]$. In either case, $\infty_K = \infty_{K'}$, $m < \infty_K$, and ∞_K is a

limit point of K. Thus K is almost closed in $J \cup \{\infty_J\}$, $g'(K) \subset h(I)$, and $g'(\infty_K) \notin h(I \cup \{\infty_I\})$. In particular. $g'(\infty_K) \neq h(\infty_I)$.

Define $g : K \cup \{\infty_K\} \to X$ such that $g|_K = g'|_K$ and $g(\infty_K) = h(\infty_I)$. Since X is a J-unique space, g is not continuous. In particular, since $g|_K$ is continuous, g fails to be continuous at the point ∞_K. Thus there exists an open neighborhood U of $h(\infty_I)$ in X such that for all $k \in K$, $(k, \infty_K]\backslash g^{-1}(U) \neq \emptyset$. Since $g(\infty_K) \in U$, $(k, \infty_K]\backslash g^{-1}(U) = (k, \infty_K)\backslash(g')^{-1}(U)$. Since $g'|_K$ is continuous $(g')^{-1}(U)$ is open in K. Let $V = (g')^{-1}(U)$ and $K'' = K\backslash V$.

By construction, K'' is closed and cofinal in K and $\infty_K = \infty_{K''}$. Moreover, $g'(K'') \subset h(I)$, $g'(K'') \cap U = \emptyset$, and $g'(\infty_K) \notin h(I \cup \{\infty_I\})$. Since h is continuous at ∞_I, there exists $M \in I$ such that if $M < i$, then $h(i) \in U$. Hence $g(K'') \subset h([0, M])$. By hypothesis $h([0, M])$ is closed in X and since $g'(\infty_K)$ is a limit point of $g'(K'')$, we have $g'(\infty_K) \in h([0, M])$. This contradicts the fact that $g'(\infty_K) \notin h(I \cup \{\infty_I\})$. $\square$

Lemma 42. *Suppose J is a non-compact, well-ordered space and X is a J-unique, J-convergence space. If $C \subset J \cup \{\infty_J\}$ is compact and $h : C \to X$ is a continuous injection, then $h(C)$ is closed in X.*

Proof. If 0 is the minimal element of J, then $\{0\}$ is a compact subspace of $J \cup \{\infty\}$. Since X is J-unique then, by Lemma 38, X is T_1 and hence, for each $x \in X$ the constant map sending $0 \to x$ is a continuous bijection onto a closed subspace of X. In particular, if $C \subset \{0\}$ then each continuous injection $h : C \to X$ has closed image.

Suppose, to obtain a contradiction, that there exists a compact subspace $C \subset J \cup \{\infty\}$ and a continuous injection $h : C \to X$ such that $h(C)$ is not closed in X. Obtain $m \in J \cup \{\infty_J\}$ minimal such that there exists a compact subspace $C \subset [0, m]$ and a continuous injection $h : C \to X$ such that $h(C)$ is not closed in X. Note $m - 1$ does not exist since otherwise the union of two closed sets $h([0, m-1] \cap C) \cup h(C \cap \{m\})$ would show $h(C)$ is closed. Thus m is a limit point of $J \cup \{\infty_J\}$. To see that m is a limit point of C, let $(j, m]$ be a basic open set in $J \cup \{\infty_J\}$ such that $j < m$. Since m is a limit point of J, we know $j + 1 < m$. To see that $((j, m] \cap C)\backslash\{m\} \neq \emptyset$, note if $((j, m) \cap C)\backslash\{m\} = \emptyset$, then the union of the two closed sets $h(C \cap [0, j+1]) \cup h(C \cap \{m\})$ would show $h(C)$ is closed. Thus m is a limit point of C and hence $m \in C$ (since C is a compact subspace of the Hausdorff space $J \cup \{\infty_J\}$).

Let $I = C \cap [0, m)$ and observe that I is almost closed in $J \cup \{\infty_J\}$ and $m = \infty_I$. If $K \subset I$ and K is compact, let $m_k = \max(K)$. Note $K \subset [0, m_k] \subset [0, m)$ and hence $h(K)$ is closed in X for all continuous injections $h : K \to X$. Thus, by Lemma 41, we conclude $h(I \cup \{\infty_I\})$ is closed in X. Since $I \cup \{\infty_I\} = C$, we contradict the fact that $h(C)$ is not closed in X. This proves the Lemma. $\square$

Theorem 43. *For any J-convergence space X, the following are equivalent:*

1. *X is a J-unique convergence space*
2. *X is a KC-space*

3. *X is weakly Hausdorff*

4. *The diagonal $\Delta = \{(x,x)|x \in X\}$ is closed in the J-product $X \times_J X$.*

Proof. 1 $\Rightarrow$ 2. Suppose X is J unique. To prove X is a KC space, suppose $A \subset X$ and A is not closed in X. It suffices to show that A is not compact. Obtain $I \subset J \cup \{\infty_J\}$ such that I is almost closed in $J \cup \{\infty_J\}$, and obtain a continuous injection $h : I \cup \{\infty_I\} \to X$ such that $h(I) \subset A$ and $h(\infty_I) \notin A$. By Lemma 42, h is a closed map and hence h is an embedding. In particular $h(I \cup \{\infty_I\})$ is closed in X. Let $U = X \backslash h(I \cup \{\infty_I\})$. Since h is an embedding, $h(I)$ is not compact and hence there exists an open cover $\{V_i\}$ of $h(I)$ such that $\{V_i\}$ has no finite subcover. Thus the open covering of A by the sets $\{U \cup V_i\}$ has no finite subcover and hence A is not compact. 2 $\Rightarrow$ 1 by Lemma 40.

2 $\Rightarrow$ 3 holds for general spaces since the image of a compact Hausdorff space is compact.

By Proposition 36, $X \times_J X = X \times_k X$. The remaining equivalences follow from the general theory of **CGWH** spaces. For 3 $\Rightarrow$ 2, note X is compactly generated and thus it follows from [26, Lemma 1.4] that compact sets in X are closed. 2 $\Leftrightarrow$ 4 is the content of [26, Proposition 2.14]. □

Declare "X has **unique transfinite convergence**" if for each unbounded well-ordered space J, if $J \cup \{\infty_J\}$ has the directed topology, if $f, g : J \cup \{\infty_J\} \to X$ are maps such that $f|_J = g|_J$, then $f(\infty_J) = g(\infty_J)$. The following example illustrates the difficulty of trying to promote Theorem 43 to the **TF** category.

Example 44. Let J be an uncountable, unbounded well-ordered set with the discrete topology and attach two unrelated maximal points. In particular, let $X = J \cup \{x, y\}$ with $x \neq y$, declare $J < x$ and $J < y$ and let $(j, x]$ and $(j, y]$ be basic open sets for $j \in J$. Then X is a transfinite sequential KC-space but X does *not* have unique transfinite sequential convergence.

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